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Non-Commutative Geometry, Multiscalars, and the Symbol Map [1]

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Abstract

Starting from the concept of the universal exterior algebra in non-commutative differential geometry we construct differential forms on the quantum phase-space of an arbitrary system. They bear the same natural relationship to quantum dynamics which ordinary tensor fields have with respect to classical hamiltonian dynamics.

1 Introduction

The mathematical setting of the standard hamiltonian formalism is the classical geometry of symplectic manifolds. Therefore all concepts and constructions of classical differential geometry (vectors, forms, exterior and Lie derivatives, ...) can be applied to the study of phase-spaces and the dynamics on them. After quantization most of these notions are lost because the position and momentum variables, previously coordinates of phase-space, are non-commuting operators then. In the following we address the question of what happens to these geometrical constructions at the quantum level. In particular, we shall present a construction of the quantum analogue of the exterior algebra. It combines the universal differential forms which appear in the non-commutative geometries of A.Connes [1] with the (Wigner–Weyl) symbol map[2]. The main virtue of this idea is that quantum and classical tensor fields are represented

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in a unified framework now, the former being a smooth deformation of the latter [3]. We shall see that the quantum differential forms are represented by multiscalar functions which depend on more than one phase-space argument. As a warm-up, we first discuss classical forms in a similar framework.

Let $\bigwedge_{\rm MS}^p$ denote the set of "multiscalar" functions $F_p(\phi_0, \phi_1, \cdots, \phi_p)$ which depend on $p+1$ arguments $\phi_i, i = 0, 1, \cdots, p$ and which vanish if two neighboring arguments are equal [4]. Now we define a map $\delta : \bigwedge_{\rm MS}^p \to \bigwedge_{\rm MS}^{p+1}$ by

$$\boxed{(\delta F_p)(\phi_0, \cdots, \phi_{p+1}) = \sum_{i=0}^{p+1} (-1)^i F_p\left(\phi_0, \cdots, \phi_{i-1}, \widehat{\phi}_i, \phi_{i+1}, \cdots, \phi_{p+1}\right)} \tag{1.1}$$

where the caret over ϕ_i means that this argument is omitted. The δ–operation maps a function of $p+1$ arguments onto a function of $p+2$ arguments. Remarkably enough, δ turns out to be nilpotent: $\delta^2 = 0$. Henceforth we shall refer to a function $F_p \in \bigwedge_{\rm MS}^p 1$ as a "p–form". On the direct sum $\bigwedge_{\rm MS}^* = \bigoplus_{p=0}^{\infty} \bigwedge_{\rm MS}^p$ there exists a natural product of a p–form F_p with a q–form G_q yielding a $(p+q)$–form $F_p \bullet G_q$:

$$(F_p \bullet G_q)(\phi_0, \cdots, \phi_p, \phi_{p+1}, \cdots, \phi_{p+q}) = F_p(\phi_0, \phi_1, \cdots, \phi_p)\, G_q(\phi_p, \phi_{p+1}, \cdots, \phi_{p+q}) \tag{1.2}$$

With respect to this product δ obeys the Leibniz rule $\delta(F_0 \bullet G_0) = (\delta F_0) \bullet G_0 + F_0 \bullet (\delta G_0)$.

Now we assume that the $\phi_i \equiv (\phi_i^a)$ are local coordinates on some manifold $\mathcal{M}$, and that the F_p's are smooth functions which transform as multiscalars under general coordinate transformations (diffeomorphisms) on $\mathcal{M}$. This means in particular that F_p evolves under the flow generated by some vector field $h = h^a(\phi)\partial_a, \partial_a \equiv \frac{\partial}{\partial \phi^a}$, according to $\partial_t F_p(t) = \sum_{i=0}^p \mathrm{V}(\phi_i) F_p(t)$, where $\mathrm{V}(\phi_i) = -h^a(\phi_i)\partial_a^{(i)}$ acts only on the ϕ_i-argument of F_p. Here the "time" t parametrizes points along the flow lines of the vector field h. In the following we restrict our attention to symplectic manifolds $\mathcal{M} \equiv \mathcal{M}_{2N}$ and to vector fields which are hamiltonian, i.e. we assume that (locally) $h^a(\phi) = \omega^{ab}\partial_b H(\phi)$ for some generating function H. Thus the Lie derivative, i.e., (minus) the RHS of the evolution equation, becomes $\mathcal{L}_p[H] = -\sum_{i=0}^p \omega^{ab}\partial_a H(\phi_i)\partial_b^{(i)}$.

Next we show how in the limit when the arguments of the multiscalar F_p are very "close" to each other, the generalized p–form $F_p \in \bigwedge_{\rm MS}^p$ gives rise to a conventional p–form. We set $\phi_0^a = \phi^a$, $\phi_i^a = \phi^a + \eta_i^a$, $i = 1, \cdots, p$ and expand $F_p(\phi, \phi + \eta_1, \cdots, \phi + \eta_p)$ to lowest order in η_i^a. We keep only terms in which all η_i's are different and obtain a sum of terms of the type

$$\eta_{i_1}^{a_1} \eta_{i_2}^{a_2} \cdots \eta_{i_l}^{a_l}\ \partial_{a_1}^{(i_1)} \cdots \partial_{a_l}^{(i_l)} F_p(\phi, \cdots, \phi) \tag{1.3}$$

for $0 \leq l \leq p$. After "stripping off" the η's, the quantities $\partial_{a_1}^{(i_1)} \cdots \partial_{a_l}^{(i_l)} F_p(\phi, \phi, \cdots, \phi)$, for $i_1, \cdots, i_l$ fixed, transform as the components of a covariant tensor field of rank l, because on each ϕ–argument there acts at most one derivative. Upon explicit antisymmetrization in the indices $a_1, \cdots, a_l$ we obtain the components of an l–form. Because the functions in $\wedge^p_{\mathrm{MS}}$ vanish if two neighboring arguments are equal one finds that the expansion contains only the term with the maximal rank $l = p$:

$$F_p\left(\phi, \phi + \eta_1, \cdots, \phi + \eta_p\right) = \eta_1^{a_1} \eta_2^{a_2} \cdots \eta_p^{a_p} \; \partial_{a_1}^{(1)} \cdots \partial_{a_p}^{(p)} F_p(\phi, \phi, \cdots, \phi) + O\left(\eta_i^2\right) \tag{1.4}$$

It is convenient to look at the ordinary differential forms induced by the multiscalars as the image of the so-called "classical map" [4] $\mathrm{Cl} : \bigwedge^p_{\mathrm{MS}}\left(\mathcal{M}_{2N}\right) \to \bigwedge^p\left(\mathcal{M}_{2N}\right)$ which is defined by

$$\left[\mathrm{Cl}\left(F_p\right)\right](\phi) = \partial_{a_1}^{(1)} \cdots \partial_{a_p}^{(p)} F(\phi, \cdots, \phi)\, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p} \tag{1.5}$$

Under the classical map the various operations defined for multiscalars are mapped onto their counterparts for ordinary differential forms:

$$\mathrm{Cl}\left(F_p \bullet G_q\right) = \mathrm{Cl}(F_p) \wedge \mathrm{Cl}(G_q),\;\; \mathrm{Cl}\left(\delta F_p\right) = d\, \mathrm{Cl}(F_p),\;\; \mathrm{Cl}\left(\mathcal{L}_p[H] F_p\right) = l_h \mathrm{Cl}(F_p) \tag{1.6}$$

2 Universal Differential Forms

Let us briefly review some properties of the universal differential forms in non–commutative geometry [1, 4]. To any algebra A we can associate its universal differential envelope ΩA, the algebra of "universal differential forms". To each element $a \in A$ we associate a new object δa. As a vector space, ΩA is defined to be the linear space of words built from the symbols $a_i \in A$ and δa_i, e.g., $a_1 \delta a_2 a_3 a_4 \delta a_3$. The multiplication in ΩA is defined to be associative and distributive over the addition $+$. The product of two elementary words is obtained by simply concatenating the two factors. One imposes the Leibniz rule $\delta\left(a_1 a_2\right) = \left(\delta a_1\right) a_2 + a_1 \delta a_2$. By virtue of this relation, any element of ΩA can be rewritten as a sum of monomials of the form $a_0 \delta a_1 \delta a_2 \cdots \delta a_p$ or $\delta a_1 \delta a_2 \cdots \delta a_p$. In order to put the two types of monomials on an equal footing it is convenient to add a new unit "1" to A, which is different from the unit A might have had already. We require $\delta 1 = 0$. As a consequence, we have to consider only words of the first type because the second one obtains for $a_0 = 1$ then. Finally one defines a linear operator

δ by the rules $\delta^2 = 0$ and $\delta(a_0 \delta a_1 \delta a_2 \cdots \delta a_p) = \delta a_0 \delta a_1 \delta a_2 \cdots \delta a_p$. By definition, $\Omega^p A$ is the linear span of the words $a_0 \delta a_1 \cdots \delta a_p$, referred to as "universal p–forms". Then

$$\Omega A = \bigoplus_{p=0}^{\infty} \Omega^p A, \qquad \Omega^0 A \equiv A, \tag{2.1}$$

is a graded differential algebra with the "exterior derivative"

$$\delta : \Omega^p A \longrightarrow \Omega^{p+1} A \tag{2.2}$$

The space $\Omega^p A$ can be identified with a certain subspace of the tensor product $A \otimes A \otimes \cdots \otimes A \equiv A^{\otimes(p+1)}$. Let us start with a few definitions. The associative $\otimes_A$–product of elements from $A^{\otimes(p+1)}$ with elements from $A^{\otimes(q+1)}$ yields elements in $A^{\otimes(p+q+1)}$. It is defined by

$$[a_0 \otimes \cdots \otimes a_p] \otimes_A [b_0 \otimes \cdots \otimes b_q] = a_0 \otimes \cdots \otimes a_{p-1} \otimes a_p b_0 \otimes b_1 \otimes \cdots \otimes b_q \tag{2.3}$$

where $a_p b_0$ is an ordinary algebra product. It is also convenient to introduce the linear multiplication maps m_i according to

$$\mathrm{m}_i [a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i \otimes \cdots \otimes a_p] = a_0 \otimes \cdots \otimes a_{i-1} a_i \otimes a_{i+1} \otimes \cdots \otimes a_p. \tag{2.4}$$

Then the construction of ΩA is as follows. We set $\Omega^0 A \equiv A$, and we identify $\delta a \in \Omega^1 A$ with $\delta a = 1 \otimes a - a \otimes 1 \in A \otimes A$. "Words" are formed by taking $\otimes_A$–products of a's and δa's. A generic element of $\Omega^1 A$ has the structure $a \delta b = a \otimes b - ab \otimes 1$ and is in the kernel of m_1 therefore: $\mathrm{m}_1(a \delta b) = ab - ab = 0$. More generally one defines

$$\Omega^p A \ = \ \Omega^1 A \otimes_A \Omega^1 A \otimes_A \cdots \otimes_A \Omega^1 A \tag{2.5}$$

where the product consists of p factors.

Let us assume that it is possible to enumerate the elements of A as $A = \{a_m, m \in \Im\}$ where $\Im$ is some index set. Then a generic p–form $\alpha_p \in \Omega^p A$ has an expansion

$$\alpha_p = \sum_{m_0 \cdots m_p} \alpha_{m_0 \cdots m_p} \, a_{m_0} \otimes a_{m_1} \otimes \cdots \otimes a_{m_p} \tag{2.6}$$

in which the coefficients $\alpha_{m_0 \cdots m_p}$ are subject to the constraints which follow from $\mathrm{m}_i \alpha_p = 0$. In this language, the action of the map δ is given by [3]

$$\boxed{\delta \alpha_p = \sum_{i=0}^{p+1} (-1)^i \sum_{m_0 \cdots m_p} \alpha_{m_0 \cdots m_p} \, a_{m_0} \otimes a_{m_1} \otimes \cdots \otimes a_{m_{i-1}} \otimes 1 \otimes a_{m_i} \otimes \cdots \otimes a_{m_p}} \tag{2.7}$$

3 Quantum Forms on Phase–Space

Let us recall some elements of the phase–space formulation of quantum mechanics [2]. We consider a set of operators $\hat{f}, \hat{g}, \cdots$ on some Hilbert space $\mathcal{H}$, and we set up a one–to–one correspondence between these operators and the complex–valued functions $f, g, \cdots \in \text{Fun}(\mathcal{M})$ defined over a suitable manifold $\mathcal{M}$. We write $f = \text{symb}\ (\hat{f})$, and we refer to the function f as the symbol of the operator $\hat{f}$. The symbol map "symb" is linear and has a well–defined inverse. An important notion is the "star product" which implements the operator multiplication at the level of symbols: $\text{symb}(\hat{f}\,\hat{g}) = \text{symb}(\hat{f}) * \text{symb}(\hat{g})$. The star product is non–commutative, but associative, because "symb" provides an algebra homomorphism between the operator algebra and the symbols.

In the physical applications we have in mind, the Hilbert space $\mathcal{H}$ is the state space of a quantum mechanical system, and the manifold $\mathcal{M} \equiv \mathcal{M}_{2N}$ is the $2N$–dimensional classical phase–space of this system. Quantum mechanical operators $\hat{f}$ are represented by functions $f = f(\phi)$, where $\phi^a = (p^1, \cdots, p^N, q^1, \cdots, q^N)$, $a = 1, \cdots, 2N$ are canonical coordinates on $\mathcal{M}_{2N}$. We assume that the phase–space has the topology of $\mathbf{R}^{2N}$, and that the symplectic 2–form $\omega = \frac{1}{2}\omega_{ab} d\phi^a \wedge d\phi^b$ has constant components: $\omega_{j,N+i} = -\omega_{N+i,j} = \delta_{ij}$. The inverse of this matrix, ω^{ab}, defines the Poisson bracket $\{f, g\}_{\text{pb}} = \partial_a f \omega^{ab} \partial_b g$. Specifying a symbol map means fixing an ordering prescription, because it associates a unique operator $\hat{f}(\hat{p}, \hat{q}) = \text{symb}^{-1}(f(p, q))$ to $f(p, q)$. We shall mostly use the Weyl symbol which associates the Weyl–ordered operator $\hat{f}$ to any polynomial f. The corresponding star product reads

$$(f * g)(\phi) = f(\phi) \exp\left[i\frac{\hbar}{2} \overleftarrow{\partial_a}\, \omega^{ab}\, \overrightarrow{\partial_b} \right] g(\phi) \tag{3.1}$$

The commutator with respect to this star product defines the well-known Moyal bracket: $\{f, g\}_{\text{mb}} = (f * g - g * f)/i\hbar$.

Our basic idea is to use the symbol map to establish a one–to–one correspondence between the abstract non–commutative differential forms in $\Omega^p A$ and functions of $p+1$ arguments, as well as between the various operations (δ, etc.) defined on them. First we extent the notion of a symbol to the elements of $A^{\otimes(p+1)}$ in a straightforward way:

$$[\text{symb}\,(\hat{a}_0 \otimes \cdots \otimes \hat{a}_p)]\,(\phi_0, \cdots, \phi_p) = [\text{symb}\,(\hat{a}_0)]\,(\phi_0) \cdots [\text{symb}\,(\hat{a}_p)]\,(\phi_p) \tag{3.2}$$

As a consequence, the map δ has a natural action on the generalized symbols:

$$\delta \,\mathrm{symb}\,(\alpha_p) \;=\; \mathrm{symb}\,(\delta\alpha_p) \tag{3.3}$$

For α_p's of the type (2.6) one can work out the explicit form of

$$\delta : \mathrm{Fun}\left(\mathcal{M}_{2N}^{(p+1)}\right) \longrightarrow \mathrm{Fun}\left(\mathcal{M}_{2N}^{(p+2)}\right) \tag{3.4}$$

Interestingly enough, one finds that δF_p is given by eq. (1.1), on which also our discussion of the *classical* exterior algebra was based. There remains a crucial difference however. The forms $F_p \in \bigwedge^p_{\mathrm{MS}}(\mathcal{M}_{2N})$ studied in the introduction were supposed to vanish when two adjacent arguments are equal. The symbols $F_p = \mathrm{symb}\,(\alpha_p)$, instead, obey a deformed version of this condition, namely $\mathrm{m}_i F_p = 0$. In the classical limit $\hbar \to 0$, when the star–product which is implicit in the multiplication map becomes the ordinary point–wise product of functions, the two notions coincide and the conditions are the same in both cases. We conclude that in the classical limit $\Omega^p A$ and $\bigwedge^p_{\mathrm{MS}}(\mathcal{M}_{2N})$ are equivalent:

$$\boxed{\lim_{\hbar\to 0} \,\mathrm{symb}\;(\Omega^p A) = \bigwedge\nolimits^p_{\mathrm{MS}}(\mathcal{M}_{2N})} \tag{3.5}$$

There exists also a very natural definition of a Lie derivative acting on the universal forms and their symbols. Let us fix a certain $\alpha_p \in \Omega^p A$ with an expansion of the type (2.6) and let us perform the same unitary transformation (generated by $\hat{H}$) on all factors of the tensor product. This leads to the "time" evolution

$$i\hbar\,\partial_t\,\alpha_p(t) = \sum_{j=0}^{p}\sum_{m_0\cdots m_p} \alpha_{m_0\cdots m_p}\;\hat{a}_{m_0}(t)\otimes\cdots\otimes\left[\hat{H},\hat{a}_{m_j}(t)\right]\otimes\cdots\otimes\hat{a}_{m_p}(t) \tag{3.6}$$

If we apply the Weyl symbol map to both sides of this equation we arrive at

$$-\,\partial_t F_p\,(\phi_0,\cdots,\phi_p;t) = \mathcal{L}^{\hbar}_p\,[H]\,F_p\,(\phi_0,\cdots,\phi_p;t) \tag{3.7}$$

with the "quantum deformed Lie derivative"

$$\mathcal{L}^{\hbar}_p\,[H] = -\sum_{i=0}^{p}\frac{2}{\hbar}\,H\,(\phi_i)\sin\left[\frac{\hbar}{2}\;\overleftarrow{\partial}^{(i)}_a\;\omega^{ab}\;\overrightarrow{\partial}^{(i)}_b\right] \tag{3.8}$$

where $H \equiv \mathrm{symb}\left(\hat{H}\right)$. In the limit $\hbar \to 0$, $\mathcal{L}^{\hbar}_p$ reduces to the classical Lie derivative for multiscalars. This suggests the interpretation of the symbols $\mathrm{symb}(\alpha), \alpha \in \Omega A$, as quantum

deformed multiscalars. When a classical multiscalar is subject to a canonical transformation, the hamiltonian vector field $-V_H = \omega^{ba}\partial_a H \partial_b$ acts on any of its arguments. In the non–commutative case, V_H is replaced by its Moyal analogue. The quantum Lie derivative commutes with the differential δ, and it gives rise to a closed, W_∞-type algebra [5]:

$$\left[\mathcal{L}_p^\hbar\left[H_1\right], \mathcal{L}_p^\hbar\left[H_2\right]\right] = -\mathcal{L}_p^\hbar\left[\{H_1, H_2\}_{\text{mb}}\right] \tag{3.9}$$

It is quite instructive to study the coincidence limit of the Moyal multiscalars $F_p = \text{symb}\left(\alpha_p\right)$. Differences relative to the classical discussion in the introduction occur because the pointwise product of functions on $\mathcal{M}_{2N}$ is now replaced by the star product. One of the consequences of this deformation is that, contrary to the classical multiscalars in $\bigwedge^p_{\text{MS}}(\mathcal{M}_{2N})$, the Moyal multiscalars F_p are not simply proportional to $\eta_1^{a_1}\cdots\eta_p^{a_p}$ in the coincidence limit: there are also terms with $\eta_1^{a_1}\cdots\eta_l^{a_l}, l < p$. As an example, let us look at the 2–form

$$\alpha_2 = \delta\hat{a}_0\delta\hat{a}_1 = \left[1\otimes\hat{a}_0 - \hat{a}_0\otimes 1\right]\otimes_A\left[1\otimes\hat{a}_1 - \hat{a}_1\otimes 1\right] \tag{3.10}$$

By applying the symbol map and expanding the arguments we are led to

$$\begin{aligned} F_2\left(\phi, \phi+\eta_1, \phi+\eta_2\right) &= \left(a_0 a_1 - a_0 * a_1\right)(\phi) + \eta_1^b\partial_b\left(a_0 a_1 - a_0 * a_1\right)(\phi) \\ &\quad + \eta_1^b\eta_2^c\,\partial_b a_0(\phi)\,\partial_c a_1(\phi) + O\left(\eta_1^2, \eta_2^2\right) \end{aligned} \tag{3.11}$$

This corresponds to an inhomogeneous differential form at the classical level. The term proportional to $\eta_1\eta_2$ is the expected 2–form evaluated on the vectors η_1 and η_2, but there is also a term linear in η (1–form) and a constant piece (0–form). For the quantum analogue of the symplectic 2–form, $\hat{\omega}_q = \omega_{ab}\,\delta\hat{\phi}^a\delta\hat{\phi}^b$, this entails that its symbol

$$\omega_q\left(\phi_0, \phi_1, \phi_2\right) = \omega_{ab}\left[\phi_1^a\phi_2^b - \phi_0^a\phi_2^b + \phi_0^a\phi_1^b\right] + iN\hbar \tag{3.12}$$

consists of the classical piece (precisely the symplectic area of the parallelogram with vertices ϕ_0, ϕ_1 and ϕ_2) augmented by a purely imaginary quantum correction $iN\hbar$. Since, for $N = 1$, $\hat{\omega}_q$ is the volume form, this term can be related to the fact that the "quantum volume" is bounded below, i.e., that quantum states cannot be localized in a phase-space volume smaller than $(2\pi\hbar)^N$. For a more detailed discussion of these issues we have to refer to [3].

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